

Random walk weakly attracted to a wall

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Abstract: We consider a random walk X_n in \mathbb{Z}_+ , starting at $X_0 = x \geq 0$, with transition probabilities

$$\mathbb{P}(X_{n+1} = X_n \pm 1 | X_n = y \geq 1) = \frac{1}{2} \mp \frac{\delta}{4y + 2\delta}$$

and $X_{n+1} = 1$ whenever $X_n = 0$. We prove $\mathbb{E}X_n \sim \text{const. } n^{1-\frac{\delta}{2}}$ as $n \nearrow \infty$ when $\delta \in (1, 2)$. The proof is based upon the Karlin-McGregor spectral representation, which is made explicit for this random walk.

KEYWORDS: Random walk, orthogonal polynomials, pinning, wetting

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1. Introduction

Random walks have been used in many different fields of physics, economics, biology... Usually, it evolves in a translation invariant environment or a random environment whose average is translation invariant. Random walks in inhomogeneous environment, in particular with a reflecting or attracting or repelling wall, have been used to mimic the behaviour of a liquid interface on top of a solid substrate, *wetting* phenomena, and a variety of other phenomena associated with the *pinning* of an interface. This is the motivation for the model presented below, whose specific form was chosen so as to derive rigorously and explicitly the result of competition between a relatively long ranged attraction and reflection at the wall.

We consider a random walk X_n in \mathbb{Z}_+ , defined by $X_0 = x \geq 0$,

$$\begin{aligned} p_y &= \mathbb{P}(X_{n+1} = X_n + 1 | X_n = y \geq 1) = \frac{1}{2} - \frac{\delta}{4y + 2\delta} = \frac{y}{2y + \delta} \\ q_y &= \mathbb{P}(X_{n+1} = X_n - 1 | X_n = y \geq 1) = \frac{1}{2} + \frac{\delta}{4y + 2\delta} = \frac{y + \delta}{2y + \delta} \end{aligned} \quad (1.1)$$

and $X_{n+1} = 1$ whenever $X_n = 0$, i.e. $p_0 = 1$, $q_0 = 0$. Let $\mathbb{P}_x(\cdot)$ denote the corresponding probabilities, and $\mathbb{E}_x(\cdot)$ the corresponding expectation values.

The walk obeys the detailed balance condition with respect to the measure π on \mathbb{Z}_+ defined up to a multiplicative constant by

$$\pi_y = \pi_0 \prod_{z=0}^{y-1} \frac{p_z}{q_{z+1}} = \pi_0 \frac{(2y + \delta)\Gamma(\delta + 1)\Gamma(y)}{\Gamma(y + \delta + 1)}, \quad y \geq 1, \quad (1.2)$$

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which obeys $\pi_y \sim y^{-\delta}$ when $y \rightarrow \infty$. We restrict our attention to $\delta > 1$, and normalise π as a probability measure, with $\pi_0 = (\delta - 1)/(2\delta)$. The dynamics is periodic, with the odd and even components exchanged under one step of the dynamics, $\pi = (\pi_{\text{even}} + \pi_{\text{odd}})/2$, with $\pi_{\text{even}} = (2\pi_0, 0, 2\pi_2, \dots)$, and $\pi_{\text{odd}} = (0, 2\pi_1, 0, 2\pi_3, \dots)$. Starting from $X_0 = 0$, we have convergence in law $X_{2n} \rightarrow X_{\infty}^{\text{even}}$ and $X_{2n+1} \rightarrow X_{\infty}^{\text{odd}}$. The first moment exists only for $\delta > 2$, with $\mathbb{E}X_{\infty} = \mathbb{E}(X_{\infty}^{\text{even}} + X_{\infty}^{\text{odd}})/2 = \delta/(2(\delta - 2))$.

We focus our attention to $1 < \delta < 2$, so that $\mathbb{E}_x X_n \rightarrow \infty$ as $n \nearrow \infty$. For $|z| < 1$ let

$$g_e(z) = \sum_{n \text{ even}} z^n \mathbb{E}_0 X_n, \quad g_o(z) = \sum_{n \text{ odd}} z^n \mathbb{E}_0 X_n \quad (1.3)$$

Our main result is the following theorem:

Theorem 1: *Let $\delta \in (1, 2)$. Then as $z \nearrow 1$ or $n \nearrow \infty$,*

$$g_e(z) = \Gamma\left(2 - \frac{\delta}{2}\right) K_{\delta} (1 - z)^{\frac{\delta}{2} - 2} \left(1 + \mathcal{O}\left((1 - z)^{(1 - \frac{\delta}{2})(\delta - 1)}\right)\right) \quad (1.4)$$

$$\mathbb{E}_0 X_n = K_{\delta} n^{1 - \frac{\delta}{2}} \left(1 + \mathcal{O}\left(n^{-(1 - \frac{\delta}{2})(\delta - 1)}\right)\right) \quad (1.5)$$

where K_{δ} is a constant depending upon δ as defined in (4.12).

Remark 1: The constant $K_{\delta} \nearrow +\infty$ when $\delta \searrow 1$ or $\delta \nearrow 2$, but the $\mathcal{O}(\cdot)$ are not uniform in δ , the theorem says nothing about $\delta = 1$ or $\delta = 2$.

Remark 2: The odd generating function $g_o(z)$ obeys the same bounds as $g_e(z)$. For any starting point x , $\mathbb{E}_x X_n$ obeys the same bounds as $\mathbb{E}_0 X_n$.

Remark 3: (1.5) follows from (1.4) by a Tauberian theorem [F, Thm. 5 p 447]. This uses monotonicity of $\mathbb{E}_0 X_n$ for n even, which holds for any reflected random walk, as recalled below together with more specific monotonicity arguments.

Consider two walks: X_n started at x , with parameter δ , and X'_n started at x' , with parameter δ' . Assume $\delta \geq \delta'$ and $x \leq x'$ and $x' - x$ even. The two walks may be coupled, e.g. using a single random number for both walks when they meet. This implies that $\mathbb{E}_x X_n$ is for each n an increasing function of x over the even integers, and also over the odd integers, and a decreasing function of $\delta \in (-1, +\infty)$. Monotonicity in δ is an example of a general monotonicity property in the transition probabilities p_y, q_y . Then

$$\mathbb{E}_0 X_{n+2} = \mathbb{P}_0(X_2 = 0) \mathbb{E}_0 X_n + \mathbb{P}_0(X_2 = 2) \mathbb{E}_2 X_n \geq \mathbb{E}_0 X_n \quad (1.6)$$

Therefore $\mathbb{E}_0 X_n$ is an increasing function of n over the even integers, and also over the odd integers. And the same property holds for $\mathbb{E}_1 X_n$. But $\mathbb{E}_x X_n$ for $x \geq 2$ is not monotonous in n , even or odd. We have

$$\mathbb{E}(X_{n+1} - X_n | X_n = y) = -\frac{\delta}{2y + \delta}, \quad y \geq 1 \quad (1.7)$$

so that $\mathbb{E}_0 X_{2n} < \mathbb{E}_0 X_{2n-1}$. And since $\mathbb{E}_0 X_n$ is increasing over the even (or the odd) integers, we have $\mathbb{E}_0 X_{2n+1} > \mathbb{E}_0 X_{2n}$.

(1.4) will be proven in Section 4. In Section 2 we exhibit the orthogonality measure of our random walk polynomials: this is Theorem 2, our main technical result, opening the way for Theorem 1. In Section 3 we solve a differential equation for generating functions, differential equation associated with the recursion formula for random walk polynomials.

Our random walk is of a very special form, but it could be used for comparison or as input for more realistic models. For example the polymer pinning model of Alexander and Zygouras [AZ] combines i.i.d. disorder with a spatially inhomogeneous Markov chain, which could be built from our random walk.

Other aspects of this model and closely related models will be treated in a forthcoming paper [H].

2. Random walk polynomials and their orthogonality measure

Our first tool is the Karlin-McGregor representation theorem [KM]: let $L^2(\pi)$ denote the Hilbert space of complex sequences $(f_y)_{y \in \mathbb{Z}_+}$ obeying $\sum_{y=0}^{\infty} |f_y|^2 \pi_y < \infty$. Then

$$(Tf)_y = p_y f_{y+1} + q_y f_{y-1} \quad (2.1)$$

defines in $L^2(\pi)$ a self-adjoint operator T of norm less or equal to one. Let $e_0 = (1, 0, 0, \dots)$. The Karlin-McGregor representation theorem gives

$$\mathbb{P}_x(X_n = y) = \frac{\pi_y}{\pi_0} \langle T^n Q_x(T) Q_y(T) e_0, e_0 \rangle \quad (2.2)$$

where $\{Q_y(t)\}_{y \in \mathbb{Z}_+}$ is a family of polynomials of degree y in t , defined recursively by $Q_{-1} = 0$, $Q_0 = 1$, and

$$tQ_y = p_y Q_{y+1} + q_y Q_{y-1}, \quad y \geq 0 \quad (2.3)$$

giving polynomials Q_y of degree y and parity $(-1)^y$, with $Q_y(1) = 1$ for all $y \geq 0$. Using the spectral resolution $\{E_t\}$ of the self-adjoint operator T , and $d\mu(t) = d\langle E_t e_0, e_0 \rangle$, one gets

$$\mathbb{P}_x(X_n = y) = \frac{\pi_y}{\pi_0} \int_{-1}^1 d\mu(t) t^n Q_x(t) Q_y(t) \quad (2.4)$$

which implies that $\{Q_y(t)\}_{y \in \mathbb{Z}_+}$ is a family of orthogonal polynomials in the probability measure $d\mu(t)$,

$$\int_{-1}^1 d\mu(t) Q_x(t) Q_y(t) = \frac{\pi_0}{\pi_y} \delta_{x,y}, \quad x, y \geq 0 \quad (2.5)$$

Given the family $\{Q_y(t)\}_{y \in \mathbb{Z}_+}$, (2.5) characterizes a unique probability measure $d\mu$, termed the orthogonality measure of the family. Letting $n \nearrow \infty$ in (2.4) shows that [KM, pp 70-71]

$$d\mu(t) = \pi_0 \left(\delta(t-1) + \delta(t+1) \right) + d\mu^c(t) \quad (2.6)$$

where $d\mu^c(t)$ is absolutely continuous with respect to the Lebesgue measure. Indeed as $n \nearrow \infty$ with $n + y - x$ even, the LHS of (2.4) tends to $2\pi_y$, while the contribution from $d\mu^c$ to the RHS tends to zero.

Our second tool is Dette's theorem [D]: the orthogonality measure $d\mu^1$ of the first associated polynomials of a random walk is related to the orthogonality measure $d\mu^*$ of the dual random walk through

$$d\mu^1(t) = \frac{1}{q_1}(1 - t^2)d\mu^*(t) \quad (2.7)$$

The first associated polynomials Q_y^1 are defined by $Q_{-1}^1 = 0$, $Q_0^1 = 1$ and

$$tQ_y^1 = p_{y+1}Q_{y+1}^1 + q_{y+1}Q_{y-1}^1, \quad y \geq 0 \quad (2.8)$$

The dual random walk polynomials Q_y^* are defined by $p_y^* = q_y$ and $q_y^* = p_y$ for all y except $p_0^* = 1$ and $q_0^* = 0$, so that $Q_0^* = 1$, $Q_1^* = t$ and

$$tQ_y^* = p_y^*Q_{y+1}^* + q_y^*Q_{y-1}^* = q_yQ_{y+1}^* + p_yQ_{y-1}^*, \quad y \geq 1 \quad (2.9)$$

Dette's theorem may be applied starting from the dual, giving

$$d\mu^{*,1}(t) = \frac{1}{p_1}(1 - t^2)d\mu(t) \quad (2.10)$$

The first associated dual polynomials $Q_y^{*,1}$ are defined by $Q_{-1}^{*,1} = 0$, $Q_0^{*,1} = 1$ and

$$tQ_y^{*,1} = q_{y+1}Q_{y+1}^{*,1} + p_{y+1}Q_{y-1}^{*,1}, \quad y \geq 0 \quad (2.11)$$

The general definitions (2.3)(2.8)(2.9)(2.11) are made explicit by inserting our p_y 's and q_y 's, giving

$$\begin{aligned} (2y + \delta)tQ_y &= yQ_{y+1} + (y + \delta)Q_{y-1}, \quad y \geq 1, \quad Q_1(t) = t \\ (2y + 2 + \delta)tQ_y^1 &= (y + 1)Q_{y+1}^1 + (y + 1 + \delta)Q_{y-1}^1, \quad y \geq 0 \\ (2y + \delta)tQ_y^* &= (y + \delta)Q_{y+1}^* + yQ_{y-1}^*, \quad y \geq 1 \\ (2y + 2 + \delta)tQ_y^{*,1} &= (y + 1 + \delta)Q_{y+1}^{*,1} + (y + 1)Q_{y-1}^{*,1}, \quad y \geq 0 \end{aligned} \quad (2.12)$$

We thus have four distinct families of orthogonal polynomials, and the corresponding four distinct orthogonality measures μ , μ^1 , μ^* , $\mu^{*,1}$.

The third and last step is to relate our polynomials to Gegenbauer polynomials of index λ , defined by $G_{-1}^\lambda = 0$, $G_0^\lambda = 1$ and

$$(2y + 2\lambda)tG_y^\lambda = (y + 1)G_{y+1}^\lambda + (y - 1 + 2\lambda)G_{y-1}^\lambda, \quad y \geq 0 \quad (2.13)$$

It appears that $Q_y^1 = G_y^{\frac{\delta}{2}+1}$, so that $Q_y^{1,*} = G_y^{\frac{\delta}{2}+1,*}$, and also $Q_y^* = \frac{y!\Gamma(\delta)}{\Gamma(y+\delta)} G_y^{\frac{\delta}{2}}$. These in turn satisfy

$$G_y^{\frac{\delta}{2}+1,*}(t) = \frac{(y+1)!\Gamma(\delta+1)}{\Gamma(y+1+\delta)} G_y^{\frac{\delta}{2},1}(t) = \frac{(y+1)!\Gamma(\frac{\delta+3}{2})}{\Gamma(y+\frac{\delta+3}{2})} P_y^{\frac{\delta-1}{2}, \frac{\delta-1}{2}}(t; 1) \quad (2.14)$$

where $P_x^{\alpha,\beta}(t; c)$ are c -associated Jacobi polynomials, or Wimp polynomials [W]. The orthogonality measure $d\mu^{*,1}$ is therefore also the orthogonality measure of the $P_y^{\frac{\delta-1}{2}, \frac{\delta-1}{2}}(t; 1)$ polynomials, namely [W, Th. 3 p. 996]

$$d\mu^{*,1}(t) = \frac{(1-t^2)^{\frac{\delta-1}{2}}}{|F(t)|^2} dt / \text{normalisation} \quad (2.15)$$

where

$$\begin{aligned} F(t) &= {}_2F_1\left(1, 1-\delta; \frac{3-\delta}{2}; \frac{1+t}{2}\right) + K e^{i\pi \frac{\delta-1}{2}} \left(\frac{1+t}{2}\right)^{\frac{\delta-1}{2}} {}_2F_1\left(\frac{1+\delta}{2}, \frac{1-\delta}{2}; \frac{1+\delta}{2}; \frac{1+t}{2}\right) \\ &= {}_2F_1\left(1, 1-\delta; \frac{3-\delta}{2}; \frac{1+t}{2}\right) + K e^{i\pi \frac{\delta-1}{2}} \left(\frac{1-t^2}{4}\right)^{\frac{\delta-1}{2}} \\ &= -1 + \mathcal{O}\left((1-t)^{\frac{\delta-1}{2}}\right) \quad \text{as } t \rightarrow 1 \end{aligned} \quad (2.16)$$

with ${}_2F_1$ the Gauss hypergeometric function and

$$K = \frac{\Gamma(\delta)\Gamma(\frac{1-\delta}{2})}{\Gamma(\frac{\delta-1}{2})} \quad (2.17)$$

We have $F(-t) = -F(t)^*$. Then (2.6)(2.10) and (2.15) give

$$d\mu^c(t) = \frac{(1-t^2)^{\frac{\delta-3}{2}}}{|F(t)|^2} dt / \text{normalisation} \quad (2.18)$$

which yields:

Theorem 2: *The orthogonality measure of the family of polynomials defined by (2.3) with (1.1) and $1 < \delta < 2$ is the even probability measure on $[-1, 1]$ defined by*

$$d\mu(t) = \pi_0 \left(\delta(t-1) + \delta(t+1) \right) + d\mu^c(t) \quad (2.19)$$

where $\delta(\cdot)$ is the Dirac measure at 0 and $d\mu^c$ is given by (2.16)-(2.18) and $\pi_0 = \frac{\delta-1}{2\delta}$, with the normalisation $\int_{-1}^1 d\mu^c = \delta^{-1}$.

3. Generating function

Using (2.4) and $X_n \leq X_0 + n$, we have

$$\mathbb{E}_0 X_n = \sum_{y=1}^n \frac{y\pi_y}{\pi_0} \int_{-1}^1 d\mu(t) t^n Q_y(t) \quad (3.1)$$

For n even, using Q_y orthogonal to $Q_0 \equiv 1$, and for y odd Q_y also orthogonal to t^n , and then using (2.6), we have

$$\begin{aligned} \mathbb{E}_0 X_n &= \sum_{\substack{y=2 \\ \text{even}}}^n \frac{y\pi_y}{\pi_0} \int_{-1}^1 d\mu^c(t) (t^n - 1) Q_y(t) \\ &= 2 \int_0^1 d\mu^c(t) (t^n - 1) \sum_{\substack{y=0 \\ \text{even}}}^n \frac{y\pi_y}{\pi_0} Q_y(t) \end{aligned} \quad (3.2)$$

The corresponding generating function is defined as

$$\begin{aligned} g_e(z) &= \sum_{n \text{ even}} z^n \mathbb{E}_0 X_n \\ &= 2 \int_0^1 d\mu^c(t) \sum_{n \text{ even}} z^n (t^n - 1) \sum_{\substack{y=0 \\ \text{even}}}^n \frac{y\pi_y}{\pi_0} Q_y(t) \\ &= 2 \int_0^1 d\mu^c(t) \sum_{y \text{ even}} \frac{y\pi_y}{\pi_0} Q_y(t) \left(\frac{(zt)^y}{1 - z^2 t^2} - \frac{z^y}{1 - z^2} \right) \end{aligned} \quad (3.3)$$

For n odd, using Q_y orthogonal to $Q_1 \equiv t$ for $y \geq 2$, and (2.5) for $y = 1$,

$$\begin{aligned} \mathbb{E}_0 X_n &= \sum_{y=1}^n \frac{y\pi_y}{\pi_0} \int_{-1}^1 d\mu(t) (t^n - t) Q_y(t) + \frac{\pi_1}{\pi_0} \int_{-1}^1 d\mu(t) t Q_1(t) \\ &= 1 + 2 \int_0^1 d\mu^c(t) (t^n - t) \sum_{\substack{y=1 \\ \text{odd}}}^n \frac{y\pi_y}{\pi_0} Q_y(t) = \mathbb{E}_1 X_{n-1} \end{aligned} \quad (3.4)$$

and the corresponding generating function

$$\begin{aligned} g_o(z) &= \sum_{n \text{ odd}} z^n \mathbb{E}_0 X_n \\ &= \frac{z}{1 - z^2} + 2 \int_0^1 d\mu^c(t) \sum_{n \text{ odd}} z^n (t^n - t) \sum_{\substack{y=1 \\ \text{odd}}}^n \frac{y\pi_y}{\pi_0} Q_y(t) \\ &= \frac{z}{1 - z^2} + 2 \int_0^1 d\mu^c(t) \sum_{y \text{ odd}} \frac{y\pi_y}{\pi_0} Q_y(t) \left(\frac{(zt)^y}{1 - z^2 t^2} - \frac{tz^y}{1 - z^2} \right) \end{aligned} \quad (3.5)$$

The recursion (2.3)(2.12) defining the random walk polynomials gives a poor uniform bound for these polynomials, e.g. $|Q_y(t)| < 3^y$ for $t \in (-1, 1)$. We define, for $|u| < 1/3$,

$$\Psi_t(u) = \sum_{y=1}^{\infty} \frac{\pi_y}{\pi_0} Q_y(t) u^y = \Gamma(\delta + 1) \sum_{y=1}^{\infty} (2y + \delta) \frac{\Gamma(y)}{\Gamma(y + \delta + 1)} Q_y(t) u^y \quad (3.6)$$

and aim at an analytic continuation giving, for $|z| < 1$,

$$\begin{aligned} g_e(z) &= z \int_0^1 d\mu^c(t) \left[\left(\frac{t\Psi'_t(zt)}{1 - z^2 t^2} - \frac{\Psi'_t(z)}{1 - z^2} \right) - \left(\frac{t\Psi'_t(-zt)}{1 - z^2 t^2} - \frac{\Psi'_t(-z)}{1 - z^2} \right) \right] \\ g_o(z) &= \frac{z}{1 - z^2} + z \int_0^1 d\mu^c(t) t \left[\left(\frac{\Psi'_t(zt)}{1 - z^2 t^2} - \frac{\Psi'_t(z)}{1 - z^2} \right) + \left(\frac{\Psi'_t(-zt)}{1 - z^2 t^2} - \frac{\Psi'_t(-z)}{1 - z^2} \right) \right] \end{aligned} \quad (3.7)$$

The recursion (2.3)(2.12) may be converted into a first order differential equation for $\Psi_t(u)$. We first get a differential equation for

$$\Phi_t(u) = \Gamma(\delta + 1) \sum_{y=1}^{\infty} \frac{\Gamma(y)}{\Gamma(y + \delta + 1)} Q_y(t) u^y \equiv \sum_{y=1}^{\infty} H_y(t) u^y \quad (3.8)$$

with

$$H_y(t) = \frac{1}{2y + \delta} \frac{\pi_y}{\pi_0} Q_y(t) = \frac{\Gamma(\delta + 1)\Gamma(y)}{\Gamma(y + \delta + 1)} Q_y(t), \quad y \geq 1 \quad (3.9)$$

obeying, with any arbitrary value for H_0 ,

$$(2y + \delta)tH_y = (y + \delta + 1)H_{y+1} + (y - 1)H_{y-1}, \quad y \geq 2 \quad (3.10)$$

with

$$H_1(t) = \frac{t}{\delta + 1}, \quad H_2(t) = \frac{Q_2(t)}{(\delta + 1)(\delta + 2)} = \frac{(\delta + 2)t^2 - (\delta + 1)}{(\delta + 1)(\delta + 2)} \quad (3.11)$$

Multiplying (3.10) by u^y and summing over $y \geq 2$ yields

$$(1 - 2tu + u^2)\Phi'_t(u) = t - u - \delta(u^{-1} - t)\Phi_t(u) \quad (3.12)$$

with $\Phi_t(0) = 0$, whose solution is

$$\Phi_t(u) = \delta^{-1} - u^{-\delta} (1 - 2tu + u^2)^{\frac{\delta}{2}} \int_0^u dv v^{\delta-1} (1 - 2tv + v^2)^{-\frac{\delta}{2}} \quad (3.13)$$

We thus get:

Lemma 3: *The function $\psi_t(u)$ defined in (3.6) may be expressed as*

$$\Psi_t(u) = 2u\Phi'_t(u) + \delta\Phi_t(u) = -1 + \frac{\delta(1 - u^2)}{1 - 2tu + u^2} \left(\delta^{-1} - \Phi_t(u) \right) \quad (3.14)$$

where $\Phi_t(u)$ is the solution of the differential equation (3.12). It extends to an analytic function in the disc $|u| < 1$. Its derivative may be expressed as

$$\Psi'_t(u) = \frac{\delta(1-u^2)}{u(1-2tu+u^2)} - \frac{B_t(u)\delta}{u^\delta(1-2tu+u^2)^{2-\frac{\delta}{2}}} \int_0^u \frac{dv v^{\delta-1}}{(1-2tv+v^2)^{\frac{\delta}{2}}} \quad (3.15)$$

with

$$B_t(u) = 4u(1-t) - 2t(1-u)^2 + \frac{\delta(1-ut)(1-u^2)}{u} \quad (3.16)$$

4. Proof of (1.4)

The leading order and next to leading order in (3.7) as $z \nearrow 1$ are found in

$$g_1(z) = \int_0^1 d\mu_c(t) \left(\frac{zt\Psi'_t(zt)}{1-z^2t^2} - \frac{z\Psi'_t(z)}{1-z^2} \right) \quad (4.1)$$

Indeed the leading orders come from the singularity at $t = 1$ in the integral. In (3.7) there is a symmetry or anti-symmetry as $z \rightarrow -z$ and $t \rightarrow -t$ jointly, associated with the even/odd symmetries. The terms not included in (4.1) correspond to $-z \simeq -1$, not singular with $t > 0$. For $1-z \ll 1$ and $1-t \ll 1$ we have

$$\begin{aligned} \frac{t}{1-z^2t^2} - \frac{1}{1-z^2} &= -(1-t) \frac{1+z^2t^2}{(1-z^2t^2)(1-z^2)} \sim -\frac{1}{2} \frac{1-t}{1-z} \frac{1}{(1-z)+(1-t)} \\ \frac{t}{1-z^2t^2} + \frac{1}{1-z^2} &= \frac{t(1-z^2)+1-z^2t^2}{(1-z^2t^2)(1-z^2)} \sim \frac{1}{2} \frac{1}{1-z} \frac{2(1-z)+1-t}{(1-z)+(1-t)} \end{aligned} \quad (4.2)$$

(3.15)(3.16) may be written as

$$q_t(z)^2 \Psi'_t(z) = \delta \frac{1-z^2}{z} q_t(z) - B_t(z) \delta z^{-\delta} q_t(z)^{\frac{\delta}{2}} \int_0^z dv v^{\delta-1} q_t(v)^{-\frac{\delta}{2}} \quad (4.3)$$

with

$$q_t(z) = 1 - 2tz + z^2 = (1-z)^2 + 2z(1-t), \quad q'_t(z) = 2(z-t) \quad (4.4)$$

Together with (4.2), we have to estimate $\Psi'_t(zt) \pm \Psi'_t(z)$ as $z \nearrow 1$, $t \nearrow 1$. We have

$$\frac{1-z^2t^2}{zt} \sim \frac{1-z^2}{z} \left(1 + \frac{1-t}{1-z} \right) \quad (4.5)$$

$$\begin{aligned} B_t(z) &\sim 4(1-t) - 2(2-\delta)(1-z)(1-t) + 2(\delta-1)(1-z)^2 \\ B'_t(z) &\sim 2(2-\delta)(1-t) - 4(\delta-1)(1-z) \end{aligned} \quad (4.6)$$

$$B_t(zt) \sim B_t(z) \left(1 - (1-t) \frac{B'_t(z)}{B_t(z)} \right)$$

$$\begin{aligned}
\int_0^1 \frac{dv v^{\delta-1}}{q_t(v)^{\frac{\delta}{2}}} &= \int_0^1 \frac{dv}{((1-v)^2 + 2(1-t))^{\frac{\delta}{2}}} + \int_0^1 dv \left[\frac{v^{\delta-1}}{q_t(v)^{\frac{\delta}{2}}} - \frac{1}{((1-v)^2 + 2(1-t))^{\frac{\delta}{2}}} \right] \\
&= (1-t)^{\frac{1-\delta}{2}} \int_0^{(1-t)^{-\frac{1}{2}}} \frac{dx}{(x^2 + 2)^{\frac{\delta}{2}}} + \mathcal{O}(1) \\
&= (1-t)^{\frac{1-\delta}{2}} \frac{2^{-\frac{1+\delta}{2}} \sqrt{\pi} \Gamma(\frac{\delta-1}{2})}{\Gamma(\frac{\delta}{2})} + \mathcal{O}(1)
\end{aligned} \tag{4.7}$$

The range $(0, 1)$ of the integral in (4.1) is split into four intervals, according to

$$0 < 1 - (1-z)^\alpha < 1 - (1-z)^\beta < 1 - (1-z)^\gamma < 1 \tag{4.8}$$

with $0 < \alpha < 1 < \beta < 2 < \gamma$ and respective contributions denoted $g_1^\alpha, g_1^{\alpha\beta}, g_1^{\beta\gamma}, g_1^\gamma$. We begin with the leading contribution, $g_1^{\alpha\beta}$:

- $1 - (1-z)^\alpha < t < 1 - (1-z)^\beta$:

$q_t(z) \sim 2(1-t)$ and $B_t(z) \sim 4(1-t)$. Then (4.3) with (4.7) yields

$$(1-t)^2 \Psi'_t(z) \sim \delta(1-z)(1-t) - (1-t)^{\frac{3}{2}} \delta \sqrt{\frac{\pi}{2}} \frac{\Gamma(\frac{\delta-1}{2})}{\Gamma(\frac{\delta}{2})} \tag{4.9}$$

and

$$\begin{aligned}
\int_{1-(1-z)^\alpha}^{1-(1-z)^\beta} dt (1-t^2)^{\frac{\delta-3}{2}} \left(\frac{t}{1-z^2 t^2} - \frac{1}{1-z^2} \right) \Psi'_t(z) &\sim \\
&\sim 2^{\frac{\delta-5}{2}} \delta \sqrt{\frac{\pi}{2}} \frac{\Gamma(\frac{\delta-1}{2})}{\Gamma(\frac{\delta}{2})} \int_{1-(1-z)^\alpha}^{1-(1-z)^\beta} dt (1-t)^{\frac{\delta}{2}-2} \frac{1-t}{1-z} \frac{1}{(1-z) + (1-t)} \\
&= 2^{\frac{\delta-5}{2}} \delta \sqrt{\frac{\pi}{2}} \frac{\Gamma(\frac{\delta-1}{2})}{\Gamma(\frac{\delta}{2})} (1-z)^{\frac{\delta}{2}-2} \int_{(1-z)^{\beta-1}}^{(1-z)^{\alpha-1}} dx \frac{x^{\frac{\delta}{2}-1}}{1+x} \\
&= 2^{\frac{\delta-5}{2}} \delta \sqrt{\frac{\pi}{2}} \Gamma\left(\frac{\delta-1}{2}\right) \Gamma\left(1-\frac{\delta}{2}\right) (1-z)^{\frac{\delta}{2}-2} \left(1 + \mathcal{O}\left((1-z)^{(1-\alpha)(1-\frac{\delta}{2})} + (1-z)^{(\beta-1)\frac{\delta}{2}}\right) \right)
\end{aligned} \tag{4.10}$$

Collecting all previous error terms we get

$$\begin{aligned}
g_1^{\alpha\beta}(z) &= \Gamma\left(2 - \frac{\delta}{2}\right) K_\delta (1-z)^{\frac{\delta}{2}-2} \left[1 + \mathcal{O}\left((1-z)^{(1-\alpha)(1-\frac{\delta}{2})} + (1-z)^\alpha \frac{\delta-1}{2} + (1-z)^{\frac{\alpha}{2}} \right. \right. \\
&\quad \left. \left. + (1-z)^{(\beta-1)\frac{\delta}{2}} + (1-z)^{2-\beta} \right) \right]
\end{aligned} \tag{4.11}$$

with

$$K_\delta = 2^{\frac{\delta-5}{2}} \sqrt{\frac{\pi}{2}} \frac{\Gamma(\frac{\delta-1}{2})}{1 - \frac{\delta}{2}} \bigg/ \int_{-1}^1 \frac{(1-t^2)^{\frac{\delta-3}{2}}}{|F(t)|^2} dt \tag{4.12}$$

and $F(t)$ given by (2.16). The five error terms in $\mathcal{O}(\cdots)$ in (4.11) come respectively from:

- 1) Extending the x -integral to $+\infty$ in (4.10)
- 2) $F(t) = -1 + \mathcal{O}((1-t)^{\frac{\delta-1}{2}})$
- 3) $\Psi'_t(zt) - \Psi'_t(z) \sim \delta$, derived from (4.3)-(4.6), with the leading contribution from (4.5).
- 4) Extending the x -integral to 0 in (4.10)
- 5) $q_t(z) \sim 2(1-t)\left(1 + \frac{(1-z)^2}{2(1-t)}\right)$ and $B_t(z) \sim 4(1-t)\left(1 + (\delta-1)\frac{(1-z)^2}{2(1-t)}\right)$.

Choosing

$$\alpha = \sup_{\delta \in (1,2)} \left(\min \left\{ (1-\alpha)\left(1 - \frac{\delta}{2}\right), \alpha \frac{\delta-1}{2} \right\} \right) = 2 - \delta \quad (4.13)$$

and β within $4 - 2\sqrt{2} < \beta < 2$, e.g. $\beta = 3/2$, then brings (4.11) to (1.4). The remaining ranges in the t -integral also fall within the previous error estimates:

- $0 < t < 1 - (1-z)^\alpha$:

$$g_1^\alpha = (1-z)^{\frac{\delta}{2}-2} \mathcal{O}\left((1-z)^{(1-\alpha)(1-\frac{\delta}{2})} + (1-z)^{\frac{\alpha}{2}} + (1-z)^{\alpha \frac{\delta-1}{2}}\right) \quad (4.14)$$

- $1 - (1-z)^\beta < t < 1 - (1-z)^\gamma$:

$$g_1^{\beta\gamma} = (1-z)^{\frac{\delta}{2}-2} \mathcal{O}\left((1-z)^{(\beta-1)\frac{\delta}{2}} + (1-z)^{2-\beta}\right) \quad (4.15)$$

- $1 - (1-z)^\gamma < t < 1$:

$$g_1^\gamma = \mathcal{O}\left((1-z)^{-1}\right) \quad (4.16)$$

This completes the proof of (1.4) and Theorem 1.

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